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On the Role of Woolley's Killing Tensor in Einstein–Maxwell Theory

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Abstract

Woolley has recently discussed conditions (one of which requires that space-time admit a Killing vector) under which the energy-momentum tensor for an electromagnetic field can be expressed in a purely 'geometric' form. In this note we show that for the case when the required Killing vector is time-like, the only asymptotically flat solutions of Woolley's equations are members of the P.I.W. class recently discovered by Perjes and independently by Israel and Wilson. This shows that although the work of Woolley exhibits a novel approach to Einstein–Maxwell theory, the usefulness of his results are much diminished.

1. *Introduction*

Woolley (1973) has shown how one can relate the covariant derivative of a Killing vector (called the Killing tensor) to the electromagnetic field tensor $F_{\mu\nu}$. Because of this the energy-momentum tensor formed from the $F_{\mu\nu}$ can be expressed completely in terms of the Killing tensor. This paper will examine the consequences of such an identification.

In Section 2 the notation is introduced and a very brief derivation of the P.I.W. class of solutions is outlined. Section 3 contains a précis of Woolley's work and also the main results of this paper.

2. *The P.I.W. Metrics*

Greek indices run from 1-4 and Latin from 1-3. A stroke | denotes covariant differentation while a comma denotes partial differentiation. A star $*$ on a scalar means complex conjugate while if it appears on a tensor it refers to the dual object.

I will consider a completely general stationary space-time. Thus the metric will exhibit only one Killing vector (time-like). The metric is:

$$ds^2 = -f^{-1}\gamma_{mn}\, dx^m\, dx^n + f(dt + \omega_m\, dx^m)^2 \tag{2.1}$$

I will use the formalism of Israel & Wilson (1972) which relies heavily on a vector calculus based on the three-dimensional positive definite metric γ_{mn}. It is found convenient to introduce a vector $\boldsymbol{\tau}$ by

$$\boldsymbol{\tau} = -f^2 \nabla \times \boldsymbol{\omega} \qquad (\text{or } f^{-2}\tau^m = \gamma^{1/2}\epsilon^{mnp}\omega_{n,p}) \tag{2.2}$$

Introducing 'electric' and 'magnetic' potentials A_0, Φ by

$$F_{0n} = A_{0,n} \qquad F^{mn} = f\gamma^{-1/2}\epsilon^{mnp}\Phi_{,p} \tag{2.3}$$

The remaining components can be read from

$$F_{\alpha\beta} + iF^*_{\alpha\beta} = \{i\epsilon_{rst}\delta^r_\alpha\delta^s_\beta\gamma^{tp}f^{-1}\gamma^{1/2} + \delta^0_\alpha\delta^p_\beta - \delta^p_\alpha\delta^0_\beta + \omega_l(\delta^l_\alpha\delta^p_\beta - \delta^l_\beta\delta^p_\alpha)\} \times \{\Psi^*_{,p}\} \tag{2.4}$$

where we have introduced the complex potential $\Psi = A_0 + i\Phi$, the Maxwell equations then take the form

$$\nabla.(f^{-1}\nabla\Psi) = if^{-2}\boldsymbol{\tau}.\nabla\Psi \tag{2.5}$$

Now introduce a scalar potential ψ by

$$\boldsymbol{\tau} + i(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) = \psi \tag{2.6}$$

and the 'Ernst' potential $E = f - \Psi\Psi^* + i\psi$, the Einstein-Maxwell equations:

$$R_{\mu\nu} = 2\{g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\}, \qquad F^{\mu\nu}{}_{;\nu} = F^{*\mu\nu}{}_{;\nu} = 0 \tag{2.7}$$

take the elegant form:

$$f\nabla^2\Psi = \nabla\Psi\,.(\nabla E + 2\Psi^*\nabla\Psi) \tag{2.8}$$

$$f\nabla^2 E = \nabla E\,.(\nabla E + 2\Psi^*\nabla\Psi) \tag{2.9}$$

$$-f^2R_{mn}(\gamma) = \tfrac{1}{2}E_{(,m}E_{,n)} + \Psi E_{(,m}\Psi^*{}_{,n)} + \Psi^*E^*{}_{(,m}\Psi_{,n)} - (E+E^*)\Psi_{(,m}\Psi^*{}_{,n)} \tag{2.10}$$

where $R_{mn}(\gamma)$ is the three-dimensional Ricci tensor formed from γ_{mn}. The P.I.W. metrics are obtained by taking $E = 0$.† Thus

$$f = \Psi\Psi^*, \qquad \boldsymbol{\tau} = -i(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*), \qquad \nabla^2(\Psi^{-1}) = 0 \tag{2.11}$$

The original paper of Israel and Wilson should be consulted if the reader requires greater detail.

3. *Solutions of Woolley's Equations*

First I shall briefly run through Woolley's work. He begins by assuming that the space-time admits a Killing vector v^σ. In this paper, only the case of a time-like Killing vector is considered. Thus I can take (2.1) as the metric of space-

† This simple method of derivation was first suggested to me by Professor W. B. Bonnor.

time, with Killing vector $v^\sigma = (0, 0, 0, 1)$. Woolley then defines two electromagnetic potentials θ, ϕ by

$$\theta_{,\mu} = F_{\mu\nu} v^\nu \qquad \phi_{,\mu} = F^*{}_{\mu\nu} v^\nu \tag{3.1}$$

The tensor $\Gamma_{\mu\nu} = -\xi_{\mu\nu} - (\theta F_{\mu\nu} + \phi F^*{}_{\mu\nu})$, where $\xi_{\mu\nu} = v_{\mu;\nu}$ satisfies Maxwell's equations $\Gamma^{\mu\nu}{}_{;\nu} = \Gamma^{*\mu\nu}{}_{;\nu} = 0$. Woolley then assumes that $\Gamma_{\mu\nu}$ and $F_{\mu\nu}$ are related by a duality rotation, i.e.

$$\Gamma_{\mu\nu} = F_{\mu\nu} \cos \epsilon + F^*{}_{\mu\nu} \sin \epsilon, \qquad \epsilon = \text{const.} \tag{3.2}$$

It then follows that

$$\begin{aligned} \xi_{\mu\nu} &= aF_{\mu\nu} + bF^*_{\mu\nu} \\ a &= -(\cos \epsilon + \theta) \\ b &= -(\sin \epsilon + \phi) \end{aligned} \tag{3.3}$$

Transvecting this equation with v^ν and integrating we find

$$(\lambda - \lambda_0) = a^2 + b^2, \qquad \lambda_0 = \text{const.}, \qquad \lambda = v_\mu v^\mu$$

Thus if we accept (3.2) then it follows that the equations we have to solve are:

$$R_{\mu\nu} = -\{\xi_{\mu\sigma} \xi^\sigma{}_\nu + \xi^*{}_{\mu\sigma} \xi^{*\sigma}{}_\nu\}/(\lambda - \lambda_0) \equiv T_{\mu\nu} \tag{3.4}$$

plus Maxwell's equations

$$\nabla . (f^{-1} \nabla \Psi) = -if^{-2} \boldsymbol{\tau} . \nabla \Psi \tag{3.5}$$

and we also have

$$\lambda - \lambda_0 = a^2 + b^2 \tag{3.6}$$

From (3.1) and (2.4) it easily follows that

$$A_0 = -(\theta + \theta_0) \qquad \Phi = (\phi + \phi_0) \qquad \theta_0, \phi_0 = \text{const.} \tag{3.7}$$

choose $\theta_0 = \cos \epsilon$ and $\phi_0 = \sin \epsilon$. Demand that $f \to 1$ as $\theta, \phi \to 0$ (essentially asymptotic flatness) then this implies that we must take $\lambda_0 = 0$. Thus $(\lambda - \lambda_0) = a^2 + b^2$ becomes $f = \Psi\Psi^*$. It is fairly straightforward to show that

$$\begin{aligned} T_{00} &= -f^{-1} \{\tfrac{1}{4} f f_{,m} f_{,n} \gamma^{mn} - \tfrac{1}{4} f^5 \omega_{m,n} \gamma^{mr} \gamma^{sn} (\omega_{s,r} - \omega_{r,s})\} \\ &= -\tfrac{1}{4} \{(\nabla f)^2 + \boldsymbol{\tau} . \boldsymbol{\tau}\} \end{aligned} \tag{3.8}$$

$$T^m{}_0 = -\tfrac{1}{2} f^2 f_{,r} \omega_{s,t} (\gamma^{rs} \gamma^{mt} - \gamma^{sm} \gamma^{rt}) = \tfrac{1}{2} (\nabla f \times \boldsymbol{\tau})^m \tag{3.9}$$

$$\gamma_{pm} \gamma_{rn} T^{mn} - \gamma_{pr} T_{00} = \tfrac{1}{2} \{f_{,p} f_{,r} + \tau_p \tau_r\} \tag{3.10}$$

Equating these to the respective components of $R_{\mu\nu}$ we find:

$$3(\nabla f)^2 - 2f \nabla^2 f - 2 \boldsymbol{\tau} . \boldsymbol{\tau} = 0 \tag{3.11}$$

$$f(\nabla \times \boldsymbol{\tau}) = \nabla f \times \boldsymbol{\tau} \tag{3.12}$$

$$R_{mn}(\gamma) = 0 \tag{3.13}$$

To solve these equations let

$$\boldsymbol{\tau} = -i(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) + \mathbf{A} \tag{3.14}$$

where **A** is some real vector. Then Maxwell's equations (3.5) imply

$$\Psi^*(\Psi \nabla^2 \Psi - 2(\nabla \Psi)^2) = i\mathbf{A} . \nabla \Psi \tag{3.15}$$

while (3.11) can be written

$$(\Psi)^2 \{(\nabla \Psi^*)^2 - \tfrac{1}{2}\Psi^* \nabla^2 \Psi^*\} + (\Psi^*)^2 \{(\nabla \Psi)^2 - \tfrac{1}{2}\Psi \nabla^2 \Psi\}$$
$$- \tfrac{1}{4}\mathbf{A} . \mathbf{A} + \tfrac{1}{2} i\mathbf{A} . (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) = 0 \tag{3.16}$$

Then using equation (3.15) it follows that $\mathbf{A} . \mathbf{A} = 0$, i.e. $\mathbf{A} = 0$. It is then immediate from (3.15) that $\nabla^2(\Psi^{-1}) = 0$ while a quick calculation verifies that (3.12) is identically satisfied. Thus for the case of a time-like Killing vector the only solutions of Woolley's equations are the P.I.W. metrics.

References

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